Uncertainty quantification for the family-wise error rate in multivariate copula models

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Outline

Simultaneous test procedures in terms of p-value copulae

Asymptotic behavior of empirically calibrated multiple tests

Estimation of an unknown copula

Application: Exchange rate risks

References:

Dickhaus, T., Gierl, J. (2013):
Simultaneous test procedures in terms of p-value copulae.
CMCGS 2013 Proceedings, 75-80.

Stange, J., Bodnar, T., Dickhaus, T. (2014):
Uncertainty quantification for the family-wise error rate in multivariate copula models.
AStA Adv. Stat. Anal., online first.

Notational setup

Given: Statistical model $(\Omega, \mathcal{F}, (\mathbb{P}_\vartheta)_{\vartheta \in \Theta})$

$\mathcal{H}_m = (H_i)_{i=1,\ldots,m}$	Family of null hypotheses with $\emptyset \neq H_i \subset \Theta$ and alternatives $K_i = \Theta \setminus H_i$
$(\Omega, \mathcal{F}, (\mathbb{P}_\vartheta)_{\vartheta \in \Theta}, \mathcal{H}_m)$	multiple test problem
$\varphi = (\varphi_i : i = 1, \ldots, m)$	multiple test for $\mathcal{H}_m$

	Test decision		
Hypotheses	0	1	
true	U_m	V_m	m_0
false	T_m	S_m	m_1
	W_m	R_m	m

Local significance level

(Strong) control of the Family-Wise Error Rate (FWER):

$$\forall \vartheta \in \Theta : \text{FWER}_\vartheta(\varphi) = \mathbb{P}_\vartheta(V_m > 0) \overset{!}{\leq} \alpha$$

Bonferroni correction:

Carry out each individual test φ_i at local level $\alpha_{\text{loc.}} := \alpha/m$.
Let $I_0(\vartheta)$ denote the index set of true hypotheses in $\mathcal{H}_m$ under ϑ.

$$\begin{aligned} \text{FWER}_\vartheta(\varphi) &= \mathbb{P}_\vartheta\left(\bigcup_{i \in I_0(\vartheta)} \{\varphi_i = 1\}\right) \\ &\leq \sum_{i \in I_0(\vartheta)} \mathbb{P}_\vartheta(\{\varphi_i = 1\}) \\ &\leq m_0 \alpha_{\text{loc.}} \leq m \alpha_{\text{loc.}} = \alpha. \end{aligned}$$

Simultaneous test procedures

K. R. Gabriel (1969), Hothorn et al. (2008)

Definition:

Define the (global) intersection hypothesis by $H_0 = \bigcap_{i=1}^{m} H_i$.

Consider the extended problem $(\Omega, \mathcal{F}, (\mathbb{P}_\vartheta)_{\vartheta \in \Theta}, \mathcal{H}_{m+1})$ with $\mathcal{H}_{m+1} = \{H_i, i \in I^* := \{0, 1, \ldots, m\}\}$.

Assume real-valued test statistics $T_i, i \in I^*$, which tend to larger values under alternatives. Then we call

(a) $(\mathcal{H}_{m+1}, \mathcal{T})$ with $\mathcal{T} = \{T_i, i \in I^*\}$ a testing family.

(b) $\varphi = (\varphi_i, i \in I^*)$ a simultaneous test procedure (STP), if

$$\forall 0 \leq i \leq m : \varphi_i = \begin{cases} 1, & \text{if} \quad T_i > c_\alpha, \\ 0, & \text{if} \quad T_i \leq c_\alpha, \end{cases} \quad \text{such that}$$

$$\forall \vartheta \in H_0 : \mathbb{P}_\vartheta(\{\varphi_0 = 1\}) = \mathbb{P}_\vartheta(\{T_0 > c_\alpha\}) \leq \alpha.$$

FWER control with STPs

Assumptions (for the moment):

1. There exists a $\vartheta^* \in H_0$ which is a least favorable parameter configuration (LFC) for the FWER of the STP φ based on $T_1, \ldots, T_m$.
2. $\forall 1 \leq i \leq m : H_i : \{\theta_i(\vartheta) = \theta_i^*\}$, where $\theta : \Theta \to \Theta'$
3. $\mathcal{L}(T_i)$ is continuous under H_i with known cdf. F_i.

Exemplary model classes:

- ANOVA1: all pairs comparisons (Tukey contrasts), multiple comparisons with a control group (Dunnett contrasts)
  Assumptions 1. - 3. are fulfilled (θ: difference operator)
- Multiple association tests in contingency tables, genetic association studies
  Assumptions 1. - 3. are fulfilled, at least asymptotically (for large sample sizes)

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Copulae

Theorem: (Sklar (1959, 1996))

Let $X = (X_1, \ldots, X_m)^\top$ a random vector with values in $\mathbb{R}^m$ and with joint cdf F_X and marginal cdfs $F_{X_1}, \ldots, F_{X_m}$.

Then there exists a function $C : [0, 1]^m \rightarrow [0, 1]$ such that

$$\forall x = (x_1, \ldots, x_m)^\top \in \bar{\mathbb{R}}^m : F_X(x) = C(F_{X_1}(x_1), \ldots, F_{X_m}(x_m)).$$

If all m marginal cdfs are continuous, the copula C is unique.

Obviously, it holds:

If all $X_i, 1 \leq i \leq m$, are marginally distributed as UNI$[0, 1]$, then $F_X = C$!

p-values, distributional transforms

Under our general assumptions 1. - 3., appropriate p-values corresponding to the T_i are given by

$$\forall 1 \leq i \leq m : p_i = 1 - F_i(T_i).$$

Properties of p_i under assumptions 1. - 3.:

- $T_i > c_\alpha \iff p_i < 1 - F_i(c_\alpha)$, if F_i is strictly isotone. We may think of $\alpha_{\text{loc.}}^{(i)} := 1 - F_i(c_\alpha)$ as a multiplicity-adjusted local significance level.
- $1 - p_i$ is equal to Rüschendorf's *distributional transform*.
- Under H_i, we have $p_i \sim \text{UNI}[0, 1]$ and $1 - p_i \sim \text{UNI}[0, 1]$.

A simple calculation

Let us construct an STP φ in terms of p-values.

Due to the above, we only have to consider multiple tests of the form $\varphi = (\varphi_i : 1 \leq i \leq m)$ with $\varphi_i = \mathbf{1}_{[0,\alpha_{\text{loc.}}^{(i)})}(p_i)$.

For arbitrary $\vartheta \in \Theta$ and $\vartheta^* \in H_0$, we get:

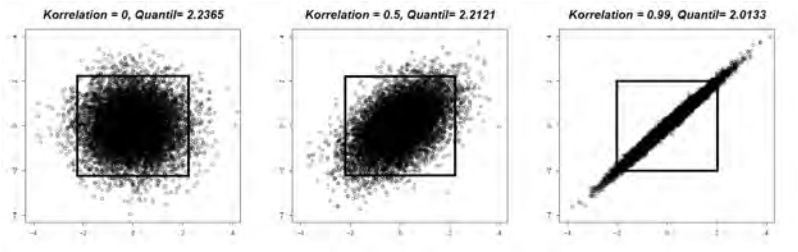
$$\begin{aligned}
\text{FWER}_\vartheta(\varphi) &= \mathbb{P}_\vartheta\left(\bigcup_{i \in I_0(\vartheta)} \{p_i < \alpha_{\text{loc.}}^{(i)}\}\right) \leq \mathbb{P}_{\vartheta^*}\left(\bigcup_{i=1}^{m} \{p_i < \alpha_{\text{loc.}}^{(i)}\}\right) \\
&= 1 - \mathbb{P}_{\vartheta^*}\left(\bigcap_{i=1}^{m} \{1 - p_i \leq 1 - \alpha_{\text{loc.}}^{(i)}\}\right) \\
&= 1 - C_{\vartheta^*}(1 - \alpha_{\text{loc.}}^{(1)}, \ldots, 1 - \alpha_{\text{loc.}}^{(m)}),
\end{aligned}$$

with C_{ϑ^*} denoting the copula of $(1 - p_i : 1 \leq i \leq m)$ under ϑ^*.

Projection method, Hothorn et al. (2008)

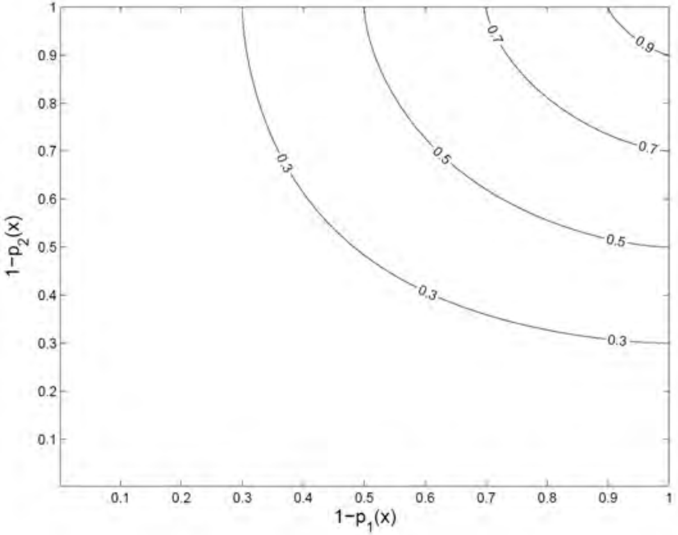
Assume that an (asymptotically) jointly normal vector of test statistics $T = (T_1, \ldots, T_m)^\top$ is at hand.

For control of the FWER by an STP based on T, determine the equicoordinate (two-sided) $(1-\alpha)$-quantile of the joint normal distribution of T and project onto the axes.



`R: vcov() + mvtnorm`

FWER control at level $\alpha = 0.3$ via contour lines of the copula C_{ϑ^*}



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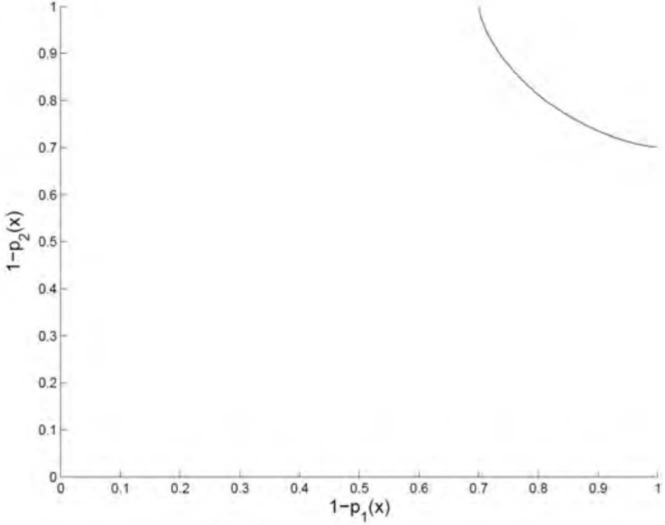

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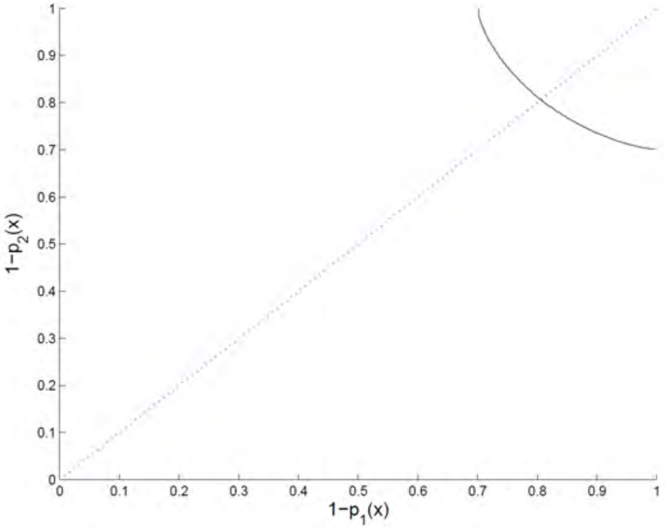

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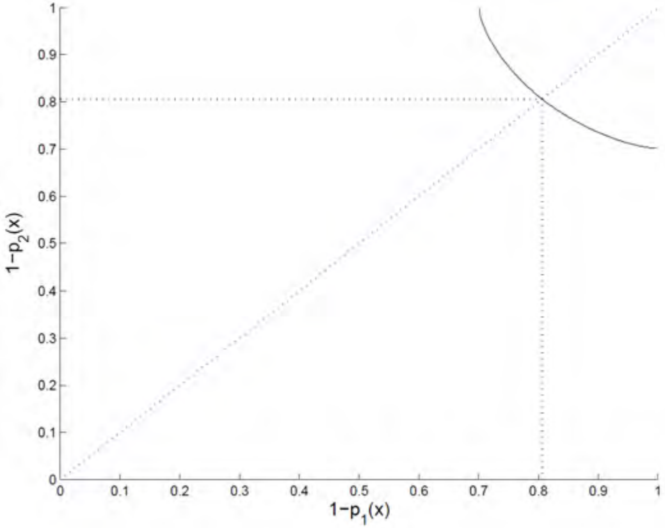

FWER control at level $\alpha = 0.3$ via contour lines of the copula C_{ϑ^*}

We obtain $\alpha_{\text{loc.}} \approx 0.2$.

Cross-check: $\Phi^{-1}(1 - \alpha_{\text{loc.}}/2)$ is equal to the tabulated normal quantile for the chosen parameters.

The structural information provided by C_{ϑ^*} increases power!

If one hypothesis is more important than the other, just change the slope of the blue straight line.

Unknown copula $C_{\vartheta*}$

In the case that we are willing to assume 1. - 3., but do not know the copula $C_{\vartheta*}$, we propose:

- Parametric copula estimation (e. g., via Spearman's ρ and/or Kendall's τ and/or Hoeffding's lemma)
- Nonparametric copula estimation (e. g., with Bernstein copulae)
- Modeling with structured (hierarchical) copulae (e. g., for block dependencies)
- Approximating contour lines by resampling or statistical learning techniques

These are research topics within our Research Unit FOR 1735 "Structural Inference in Statistics: Adaptation and Efficiency".

Extended model setup with copula parameter

Extended model for the family of probability measures:

$\mathcal{P} = (\mathbb{P}_{\vartheta,\eta} : \vartheta \in \Theta, \eta \in \Xi)$

$\vartheta \in \Theta$	Parameter of interest ($H_j \subset \Theta$, $1 \leq j \leq m$),
$\eta \in \Xi$	Nuisance (copula) parameter representing the dependency structure

Fundamental assumption: η does not depend on ϑ.

FWER control in the extended model:

$$\sup_{\vartheta \in \Theta, \eta \in \Xi} \text{FWER}_{\vartheta,\eta}(\varphi) \overset{!}{\leq} \alpha.$$

LFC $\vartheta^* \in H_0$: Put $\mathbb{P}^*_\eta = \mathbb{P}_{\vartheta^*,\eta}$ and $\text{FWER}^*_\eta(\varphi) = \text{FWER}_{\vartheta^*,\eta}(\varphi)$.

Empirical calibration of critical values

We recall for a multiple test φ with test statistics $T_1, \ldots, T_m$ and critical values $c_1, \ldots, c_m$ under our general assumptions 1. - 3.:

$$\begin{aligned} \text{FWER}_{\vartheta,\eta}(\varphi) &\leq \text{FWER}^*_\eta(\varphi) = \mathbb{P}^*_\eta\left(\bigcup_{j=1}^m \{T_j > c_j\}\right) \\ &= 1 - C_\eta(F_1(c_1), \ldots, F_m(c_m)). \end{aligned}$$

Empirical calibration of φ:

- Assume that the dependence structure of $\mathbf{T}$ is determined by the copula function C_{η_0}, $\eta_0 \in \Xi$.
- Utilization of an estimate $\hat{\eta}$ for η_0 leads to the empirically calibrated critical values $\hat{\mathbf{c}} = \mathbf{c}(\hat{\eta})$ and the calibrated test $\hat{\varphi}$.
- Calibrated local significance levels: Take $\mathbf{u}(\hat{\eta})$ from the set $C_{\hat{\eta}}^{-1}(1-\alpha)$ and put $\alpha_{\text{loc.}}^{(j)} = 1 - u_j(\hat{\eta})$, $1 \leq j \leq m$.

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Regard $\text{FWER}^*_{\eta_0}(\varphi)$ as a derived parameter of the copula model for $\mathbf{T}$.

Theorem:

Assume that $C_{\eta_0} \in \{C_\eta | \eta \in \Xi \subseteq \mathbb{R}^p\}$, $p \in \mathbb{N}$.

Suppose an estimator $\hat{\eta}_n : \Omega \to \Xi$ of η_0 fulfilling

$$\sqrt{n}(\hat{\eta}_n - \eta_0) \xrightarrow{d} \mathcal{N}_p(0, \Sigma_0) \quad \text{as} \quad n \to \infty.$$

Then, under standard regularity assumptions, it holds:

a) Asymptotic Normality (Delta method)

$$\sqrt{n}\left(\text{FWER}^*_{\eta_0}(\hat{\varphi}) - \alpha\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2_{\eta_0}).$$

b) Asymptotic Confidence Region ($\hat{\sigma}^2_n$ consistent for $\sigma^2_{\eta_0}$)

$$\lim_{n\to\infty} \mathbb{P}^*_{\eta_0}\left(\sqrt{n}\frac{\text{FWER}^*_{\eta_0}(\hat{\varphi}) - \alpha}{\hat{\sigma}_n} \leq z_{1-\delta}\right) = 1 - \delta.$$

Three "inversion formulas"

Lemma:

X and Y real-valued random variables with marginal cdfs F_X and F_Y and bivariate copula C_η, depending on a copula parameter η.

$\sigma_{X,Y}$: Covariance of X and Y
$\rho_{X,Y}$: Spearman's rank correlation coefficient (population version)
$\tau_{X,Y}$: Kendall's tau (population version)

Then it holds:

$$\begin{aligned}
\sigma_{X,Y} = f_1(\eta) &= \int_{\mathbb{R}^2} [C_\eta\{F_X(x), F_Y(y)\} \\
&\quad - F_X(x)F_Y(y)]\, dx\, dy, \\
\rho_{X,Y} = f_2(\eta) &= 12 \int_{[0,1]^2} C_\eta(u,v)\, du\, dv - 3, \\
\tau_{X,Y} = f_3(\eta) &= 4 \int_{[0,1]^2} C_\eta(u,v)\, dC_\eta(u,v) - 1.
\end{aligned}$$

Example: Gumbel-Hougaard copulae

(One-parametric Archimedean copula)

$$C_\eta(u_1, \ldots, u_m) = \exp\left(-\left[\sum_{j=1}^{m}(-\ln(u_j))^\eta\right]^{1/\eta}\right), \ \eta \geq 1.$$

Taking $m = 2$, we obtain

$$\tau_\eta = \frac{\eta - 1}{\eta}$$

and, consequently,

$$\eta = (1 - \tau)^{-1}. \qquad (1)$$

Thus, η can easily be calibrated by a method of moments (plug-in of an augmented sample version of τ into (1)).

Gumbel-Hougaard copulae and $\max$-stability

Proposition: ($\max$-stability of Gumbel-Hougaard copulae)

For all $\eta \geq 1$ and $(u_1, \ldots, u_m)^\top \in [0, 1]^m$, it holds:

1. C_η is a max-stable copula, i. e.,

$$\forall n \in \mathbb{N} : C_\eta(u_1, \ldots, u_m)^n = C_\eta(u_1^n, \ldots, u_m^n).$$

2. It exists a family of copulas such that for any member C, it holds

$$\lim_{n\to\infty} \left(C(u_1^{1/n}, \ldots, u_m^{1/n}) \right)^n = C_\eta(u_1, \ldots, u_m).$$

$\Longrightarrow$ **Applications of Gumbel-Hougaard copulae in multivariate extreme value statistics**

Example: Multiple support tests

$\mathbf{X}_1, \ldots, \mathbf{X}_n$: sample of iid. random vectors with values in $[0, \infty)^m$, each of which distributed as $\mathbf{X} = (X_1, \ldots, X_m)^\top$ with

$$\forall 1 \leq j \leq m : X_j \stackrel{d}{=} \vartheta_j Z_j, \ \vartheta_j > 0,$$

where Z_j has cdf. $F_j : [0, 1] \to [0, 1]$.
Parameter of interest: $\vartheta = (\vartheta_1, \ldots, \vartheta_m)^\top \in \Theta = (0, \infty)^m$.

Multiple test problem ($\vartheta_j^* : 1 \leq j \leq m$ given constants):

$$H_j : \{\vartheta_j \leq \vartheta_j^*\} \ \text{versus} \ K_j : \{\vartheta_j > \vartheta_j^*\}, \ j = 1, \ldots, m$$

Test statistics: $T_j = \max_{1 \leq i \leq n} X_{i,j} / \vartheta_j^*, \ 1 \leq j \leq m$

If the copula of $\mathbf{X}$ is in the domain of attraction of some C_η, our theory applies, at least asymptotically.

An application to exchange rate risks

Consider daily exchange rates:
EUR/CNY, EUR/HKD, EUR/MXN, and EUR/USD.

Data from 01/07/2010 to 30/06/2014 (http://sdw.ecb.europa.eu) were transformed into log-returns.

Entire sample was split into two sub-samples, where the first sub-sample consists of the data for the first three years.

Research question:

For which of the four time series does the tail behavior of the returns remain stable during the fourth year of analysis?

Stochastic model for extreme returns

It is common practice to model excesses over large thresholds u by generalized Pareto distributions (GPDs) with cdf

$$G_{\xi,\vartheta}(x) = \begin{cases} 1 - (1 + \xi x/\vartheta)^{-1/\xi}, & \xi \neq 0, \\ 1 - \exp(-x/\vartheta), & \xi = 0, \end{cases}$$

where $x \geq 0$ for $\xi \geq 0$ and $0 \leq x \leq -\vartheta/\xi$ if $\xi < 0$.

Table: Maximum likelihood estimates of the GPD parameters based on data from 01/07/2010 until 30/06/2013

Parameter	EUR/CNY	EUR/HKD	EUR/MXN	EUR/USD
ξ	-0.18027	-0.14824	-0.05606	-0.22055
	(0.09342)	(0.09707)	(0.10757)	(0.06810)
ϑ	0.00315	0.00309	0.00485	0.00403
	(0.00046)	(0.00046)	(0.00076)	(0.00044)
$x_0 = u - \vartheta/\xi$	0.02503	0.02868	0.09441	0.02620

Results of the data analysis on second sub-sample

Table: Lower confidence limits for ϑ_j and $x_{0,j}$, $1 \leq j \leq 4$, for the second time period from 01/07/2013 until 30/06/2014

	$\underline{\vartheta}_j$			
	EUR/CNY	EUR/HKD	EUR/MXN	EUR/USD
Bonferroni	0.002384	0.002189	0.002248	0.002691
Šidák	0.002387	0.002192	0.002253	0.002694
Gumbel $G_{\hat{\eta}}$	0.002510	0.002321	0.002449	0.002809
	$\underline{x}_{0,j}$			
	EUR/CNY	EUR/HKD	EUR/MXN	EUR/USD
Bonferroni	0.020769	0.022605	0.047982	0.020143
Šidák	0.020784	0.022625	0.048063	0.020155
Gumbel $G_{\hat{\eta}}$	0.021465	0.023501	0.051565	0.020678

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