Union-Intersection Based Goodness-of-Fit Tests in Terms of Local Levels

V. Gontscharuk and H. Finner

Institute for Biometrics and Epidemiology, German Diabetes Center at the Heinrich-Heine-University Düsseldorf, Leibniz Center for Diabetes Research

Multiple Tests and Confidence Intervals Cologne, 24. June 2015
Goodness-of-Fit Tests

\(X_1, \ldots, X_n \overset{iid}{\sim} F\) continuous, \(X_{1:n}, \ldots, X_{n:n}\) related order statistics

\[H_0^+ : F(x) \leq F_0(x) \quad \text{or} \quad H_0 : F(x) = F_0(x)\]

\(\varphi^+\) rejects \(H_0^+\) (i.e., \(\varphi^+ = 1\)) iff \(X_{i:n} \leq c_{i,n}\) for some \(i\).

\(\varphi\) rejects \(H_0\) (i.e., \(\varphi = 1\)) iff \(X_{i:n} \leq c_{i,n}\) or \(X_{i:n} \geq \tilde{c}_{i,n}\) for some \(i\).

**Assumption:** \(0 \leq c_{1,n} < \ldots < c_{n,n} < 1, 0 < \tilde{c}_{1,n} < \ldots < \tilde{c}_{n,n} \leq 1\) and \(c_{i,n} < \tilde{c}_{i,n}, i = 1, \ldots, n\).

**Examples:** Kolmogorov-Smirnov, Anderson-Darling, Berk-Jones tests and other tests based on \(\varphi\)-divergences
Goodness-of-Fit Tests

\[ X_1, \ldots, X_n \overset{iid}{\sim} F \text{ continuous, } X_{1:n}, \ldots, X_{n:n} \text{ related order statistics} \]

\[ H_0^+ : F(x) \leq x \text{ or } H_0 : F(x) = x \]

\( \varphi^+ \) rejects \( H_0^+ \) (i.e., \( \varphi^+ = 1 \)) iff \( X_{i:n} \leq c_{i,n} \) for some \( i \).

\( \varphi \) rejects \( H_0 \) (i.e., \( \varphi = 1 \)) iff \( X_{i:n} \leq c_{i,n} \) or \( X_{i:n} \geq \tilde{c}_{i,n} \) for some \( i \).

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The Union-Intersection Principle and Local Levels

\[ U_1, \ldots, U_n \overset{iid}{\sim} U(0, 1), \ U_{1:n}, \ldots, U_{n:n} \text{ related order statistics} \]

\[ H_i^+ \text{ (or } H_i) \text{ is true if } X_{i:n} \geq U_{i:n} \text{ (or } X_{i:n} \overset{D}{=} U_{i:n}) \]

\[ \Rightarrow H_0^+ \subseteq \cap_{i=1}^n H_i^+ \text{ and } H_0 \subseteq \cap_{i=1}^n H_i \]

\[ \varphi_i^+ \text{ rejects } H_i^+ \text{ (i.e., } \varphi_i^+ = 1) \text{ iff } X_{i:n} \leq c_{i,n}. \]

\[ \varphi_i \text{ rejects } H_i \text{ (i.e., } \varphi_i = 1) \text{ iff } X_{i:n} \leq c_{i,n} \text{ or } X_{i:n} \geq \tilde{c}_{i,n}. \]

\[ \Rightarrow \varphi^+ = \max_i \varphi_i^+ \text{ and } \varphi = \max_i \varphi_i \text{ are union-intersection tests.} \]

Local levels:

\[ \alpha_{i,n}^+ = \mathbb{P}(\varphi_i^+ = 1|H_0) = \mathbb{P}(U_{i:n} \leq c_{i,n}) \]

\[ \alpha_{i,n} = \mathbb{P}(\varphi_i = 1|H_0) = \mathbb{P}(U_{i:n} \leq c_{i,n}) + \mathbb{P}(U_{i:n} \geq \tilde{c}_{i,n}) \]
Local Levels of Kolmogorov-Smirnov Tests

\[ \alpha_{i,n}^+ \text{ for } \alpha = 0.05 \text{ and } n = 100, 500, 1000 \text{ and the corresponding asymptotic local levels (black curve)} \]
Local Levels of Tests Based on $\varphi$-Divergences

$$\alpha_{i,n} \equiv \alpha_{i,n}(s) \text{ with } \alpha = 0.05 \text{ and } n = 1000$$

$s = 2$: Higher Criticism (HC), $s = 1$: Berk-Jones (BJ),

$s = 0$: reversed BJ, $s = -1$: studentized HC
GOF Tests in Terms of Local Levels

For given critical values $c_{i,n}, \tilde{c}_{i,n}, i = 1, \ldots, n$, we get

$$\alpha_{i,n}^+ = \mathbb{P}(U_{i:n} \leq c_{i,n}) \quad \text{and} \quad \alpha_{i,n} = \mathbb{P}(U_{i:n} \leq c_{i,n}) + \mathbb{P}(U_{i:n} \geq \tilde{c}_{i,n}).$$

$$U_{i:n} \sim \text{Beta}(i, n - i + 1) \quad \text{with cdf} \quad F_{i,n-i+1}$$

**One-sided test:** $c_{i,n} = F_{i,n-i+1}^{-1}(\alpha_{i,n}^+), i = 1, \ldots, n$

**Two-sided test:** Split $\alpha_{i,n}$ in $\alpha_{i,n}^{(1)}$ and $\alpha_{i,n}^{(2)}$ such that

$\alpha_{i,n} = \alpha_{i,n}^{(1)} + \alpha_{i,n}^{(2)}$ (e.g., $\alpha_{i,n}^{(1)} = \alpha_{i,n}^{(2)} = \alpha_{i,n}/2$)

$\Rightarrow \quad c_{i,n} = F_{i,n-i+1}^{-1}(\alpha_{i,n}^{(1)}) \quad \text{and} \quad \tilde{c}_{i,n} = F_{i,n-i+1}^{-1}(1 - \alpha_{i,n}^{(2)}), i = 1, \ldots, n$
GOF Test With Equal Local Levels

\[ \alpha_{1,n} = \ldots = \alpha_{n,n} = \alpha_{n}^{\text{loc}} \quad \text{or} \quad \alpha_{1,n} = \ldots = \alpha_{n,n} = \alpha_{n}^{\text{loc}} \]

\[ \varphi^{+}(\alpha_{n}^{\text{loc}}) : c_{i,n} = F_{i,n-i+1}^{-1}(\alpha_{n}^{\text{loc}}) \]

\[ \varphi(\alpha_{n}^{\text{loc}}) : c_{i,n} = F_{i,n-i+1}^{-1}(\alpha_{n}^{\text{loc}}/2) \quad \text{and} \quad \tilde{c}_{i,n} = 1 - F_{i,n-i+1}^{-1}(\alpha_{n}^{\text{loc}}/2) \]

\[ p_{i,n} = F_{i,n-i+1}(X_{i:n}) \quad i = 1, \ldots, n, \quad \text{one-sided } p\text{-values} \]

\[ M_{n}^{+} = \min_{i=1,\ldots,n} p_{i,n} \quad \text{and} \quad M_{n} = 2 \min_{i=1,\ldots,n} \{p_{i,n}, 1 - p_{i,n}\} \]

\[ \varphi^{+}(\alpha_{n}^{\text{loc}}) = 1 \quad \text{iff} \quad M_{n}^{+} \leq \alpha_{n}^{\text{loc}} \quad \text{and} \quad \varphi(\alpha_{n}^{\text{loc}}) = 1 \quad \text{iff} \quad M_{n} \leq \alpha_{n}^{\text{loc}} \]

\[ \Rightarrow \varphi^{+}(\alpha_{n}^{\text{loc}}) \quad \text{and} \quad \varphi(\alpha_{n}^{\text{loc}}) \quad \text{are minimum } p\text{-value (minP) GOF tests.} \]
minP Tests: Different Names and Representations

- Berk & Jones [1978, 1979] introduced $M_n^+$ and $M_n$ as minimum level attained statistics: optimality and Bahadur efficiency;
- Buja & Rolke [2006] (unpublished): minP tests in terms of bounding functions;
- Gontscharuk, Landwehr & Finner (talks at MCP 2011 and MCP 2013): GOF tests with equal local levels, HC tests;
- Aldor et al. [2013]: tests based on the new tail-sensitive simultaneous confidence bands;
- Mary & Ferrari [2014]: non-asymptotic standardization of binomial counts, HC framework;
- Preprints: calibrated KS tests in Moskovitch et al., Dirichlet-based tests in Kaplan & Goldman.
Level $\alpha$ minP GOF Tests

Find $\alpha_n^{loc}$ so that $P(M_n^+ \leq \alpha_n^{loc} | H_0) = \alpha$ or $P(M_n \leq \alpha_n^{loc} | H_0) = \alpha$.

Let $\varphi$ be an exact level $\alpha$ test with local levels $\alpha_{i,n}$. Then

$$\min_{i=1,\ldots,n} \alpha_{i,n} \leq \alpha_n^{loc} \leq \max_{i=1,\ldots,n} \alpha_{i,n}.$$ 

Example: $i \equiv i_n$ such that $i/n \to \zeta \in (0, 1)$ leads to the asymptotic KS local level $\alpha_{\zeta}^{KS} = 1 - \Phi(-\log(\alpha)/(2\zeta(1 - \zeta)))$. Hence,

$$\alpha_n^{loc} \leq 1 - \Phi(\sqrt{-2 \log(\alpha)}) + o(1), \quad n \in \mathbb{N}.$$
GOF Tests Based on $\phi$-Divergences

$$\alpha_{i,n} \equiv \alpha_{i,n}(s) \text{ with } \alpha = 0.05 \text{ and } n = 1000,$$

$s = 2$: Higher Criticism, $s = 1$: Berk-Jones
Higher Criticism (HC) Statistics

- HC statistics are normalized KS statistics:
  \[
  HC_n^+ = \max_{i=1,\ldots,n} \sqrt{n} \frac{i/n - X_{i:n}}{\sqrt{X_{i:n}(1 - X_{i:n})}},
  \]
  \[
  HC_n = \max_{i=1,\ldots,n} \left\{ \sqrt{n} \frac{i/n - X_{i:n}}{\sqrt{X_{i:n}(1 - X_{i:n})}}, \sqrt{n} \frac{X_{i:n} - (i - 1)/n}{\sqrt{X_{i:n}(1 - X_{i:n})}} \right\}.
  \]

- Eicker [1979] and Jaeschke [1979] provided a lot of asymptotic results.

- Local levels of one-sided HC asymptotic level \( \alpha \) tests
  \[
  \alpha_{i,n}^{HC}(\alpha) \approx \frac{-\log(1 - \alpha)}{2 \log_2(n) \log(n)}
  \]
  for the most \( i \in \{1, \ldots, n\} \), cf. Gontscharuk et al.[2015]

HC local levels are asymptotically equal in the sensitivity range.
Theorem 1: (Gontscharuk & Finner [2015]) The minP test with critical value $d_n$ is an asymptotic level $\alpha$ test iff

$$\lim_{n \to \infty} \frac{d_n}{\alpha_n^*} = 1 \quad \text{with} \quad \alpha_n^* \equiv \alpha_n^*(\alpha) = -\frac{\log(1 - \alpha)}{2 \log_2(n) \log(n)}.$$

Remark:

- Critical values $d_n$ related to asymptotic level $\alpha$ minP tests converge to 0 for $n \to \infty$.
- The asymptotic critical value is the same for one- and two-sided minP tests.
\textbf{Theorem 2}: (Gontscharuk \& Finner [2015]) The \( z \)-transformed minP statistics \( \Phi^{-1}(1 - M_n^+) \) and \( \Phi^{-1}(1 - M_n/2) \) have the same asymptotic distribution as \( HC_n^+ \) and \( HC_n \), resp.

\textbf{Remark}: Theorem 2 implies that the minP critical values

\[ \alpha'_n = 1 - \Phi(b_n(t_\alpha^+)) \quad \text{and} \quad \alpha''_n = 2(1 - \Phi(b_n(t_\alpha))) , \]

where \( b_n(t) = \sqrt{2 \log_2(n)} + (\log_3(n) - \log(\pi) + 2t)/(2\sqrt{2 \log_2(n)}) \), \( t_\alpha^+ = -\log(-\log(1 - \alpha)) \) and \( t_\alpha = -\log(-\log(1 - \alpha)/2) \), lead to asymptotic level \( \alpha \) minP tests.
Applicability of Asymptotic Results

**Left graph:** $\alpha_n^*, \alpha_n', \alpha_n''$, $\alpha_n^{loc}$ related to level $\alpha$ one-sided (upper curves) and two-sided (lower curves) minP tests.

**Right graph:** $\mathbb{P}(M_n^+ \leq d_n|H_0)$ (lower curves) and $\mathbb{P}(M_n \leq d_n|H_0)$ (upper curves) for $d_n = \alpha_n^*, \alpha_n', \alpha_n''$
Finite Approximation for $\alpha_{n}^{loc}$

The asymptotic HC local levels are given by

$$\alpha_{i,n}^{HC}(\alpha) = \frac{-\log(1 - \alpha)}{2 \log_2(n) \log(n)} \left[ 1 + O\left(\frac{\log_3(n)}{\log_2(n)}\right) \right]$$

for $\log(n) \leq i \leq n - \log(n)$, cf. Gontscharuk et al. [2015].

Define

$$d_n(\alpha) = \frac{-\log(1 - \alpha)}{2 \log_2(n) \log(n)} \left[ 1 - c_{\alpha} \frac{\log_3(n)}{\log_2(n)} \right],$$

where $c_{\alpha} \in \mathbb{R}$ is a suitable constant.

Since $d_n(\alpha)/\alpha_n^{*} \rightarrow 1$ as $n \rightarrow \infty$, Theorem 1 implies that $d_n(\alpha)$ leads to asymptotic level $\alpha$ minP tests.
Approximated And Exact Critical Values

Critical values related to two-sided level $\alpha$ minP GOF tests

\[ d_n(\alpha) \text{ with } c_\alpha = 1.6, 1.3, 1.1 \text{ (diamonds from bottom to top)} \] and
\[ \alpha_n^{loc} \text{ for } \alpha = 0.01, 0.05, 0.1 \text{ (solid curves from bottom to top)} \]
Simulated Global Levels Related to $d_n(\alpha)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha = 0.01$</th>
<th>$\alpha = 0.05$</th>
<th>$\alpha = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 10^4$</td>
<td>$0.00966$ (0.00972)</td>
<td>$0.04874$ (0.04905)</td>
<td>$0.09969$ (0.09937)</td>
</tr>
<tr>
<td>$n = 5 \times 10^4$</td>
<td>$0.00961$</td>
<td>$0.04971$</td>
<td>$0.10016$</td>
</tr>
<tr>
<td>$n = 10^5$</td>
<td>$0.01018$</td>
<td>$0.05019$</td>
<td>$0.10188$</td>
</tr>
<tr>
<td>$n = 5 \times 10^5$</td>
<td>$0.01001$</td>
<td>$0.05018$</td>
<td>$0.10115$</td>
</tr>
<tr>
<td>$n = 10^6$</td>
<td>$0.00973$</td>
<td>$0.04942$</td>
<td>$0.10135$</td>
</tr>
</tbody>
</table>

$\mathbb{P}(M_n \leq d_n(\alpha)|H_0)$ (and $\mathbb{P}(M_n \leq \alpha_{n}^{loc}|H_0)$ for $n = 10^4$ only) simulated by $10^5$ repetitions, where $d_n(\alpha)$ is based on $c_{\alpha} = 1.6, 1.3, 1.1$ for $\alpha = 0.01, 0.05, 0.1$, resp.

The minP GOF test works very well at least for considered $\alpha$- and $n$-values.
Take Home Message

- Local levels can be seen as a measure of local sensitivity.
- One may construct a tailored GOF test by means of local levels.
- The minP test is a test with equal local levels.
- HC asymptotics is the key to the minP asymptotics.
- We provide three competing critical values leading to the asymptotic level $\alpha$ minP tests.
- The minP (as well as HC) asymptotics is very slow.
- New approximation for the minP critical value that works well for finite samples and leads to asymptotic level $\alpha$ tests.
References