Union-Intersection Based Goodness-of-Fit Tests in Terms of Local Levels

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Goodness-of-Fit Tests

$X_1, \ldots, X_n \overset{iid}{\sim} F$ continuous, $X_{1:n}, \ldots, X_{n:n}$ related order statistics

$$H_0^+ : F(x) \leq F_0(x) \text{ or } H_0 : F(x) = F_0(x)$$

φ^+ rejects H_0^+ (i.e., $\varphi^+ = 1$) iff $X_{i:n} \leq c_{i,n}$ for some i.

φ rejects H_0 (i.e., $\varphi = 1$) iff $X_{i:n} \leq c_{i,n}$ or $X_{i:n} \geq \tilde{c}_{i,n}$ for some i.

Assumption: $0 \leq c_{1,n} < \ldots < c_{n,n} < 1$, $0 < \tilde{c}_{1,n} < \ldots < \tilde{c}_{n,n} \leq 1$ and $c_{i,n} < \tilde{c}_{i,n}$, $i = 1, \ldots, n$.

Examples: Kolmogorov-Smirnov, Anderson-Darling, Berk-Jones tests and other tests based on φ-divergences

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The Union-Intersection Principle and Local Levels

$U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$, $U_{1:n}, \dots, U_{n:n}$ related order statistics

H_i^+ (or H_i) is true if $X_{i:n} \overset{st.}{\geq} U_{i:n}$ (or $X_{i:n} \overset{\mathcal{D}}{=} U_{i:n}$)

$\Rightarrow$ $H_0^+ \subseteq \cap_{i=1}^n H_i^+$ and $H_0 \subseteq \cap_{i=1}^n H_i$

φ_i^+ rejects H_i^+ (i.e., $\varphi_i^+ = 1$) iff $X_{i:n} \leq c_{i,n}$.
φ_i rejects H_i (i.e., $\varphi_i = 1$) iff $X_{i:n} \leq c_{i,n}$ or $X_{i:n} \geq \tilde{c}_{i,n}$.

$\Rightarrow$ $\varphi^+ = \max_i \varphi_i^+$ and $\varphi = \max_i \varphi_i$ are union-intersection tests.

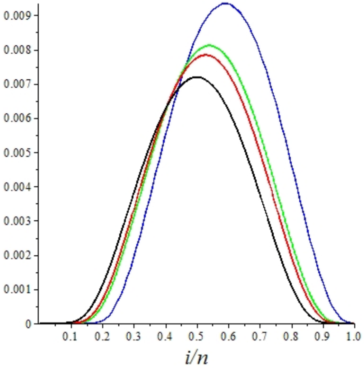
Local levels:

$$\boxed{\alpha_{i,n}^+ = \mathbb{P}(\varphi_i^+ = 1 | H_0) = \mathbb{P}(U_{i:n} \leq c_{i,n})}$$

$$\boxed{\alpha_{i,n} = \mathbb{P}(\varphi_i = 1 | H_0) = \mathbb{P}(U_{i:n} \leq c_{i,n}) + \mathbb{P}(U_{i:n} \geq \tilde{c}_{i,n})}$$

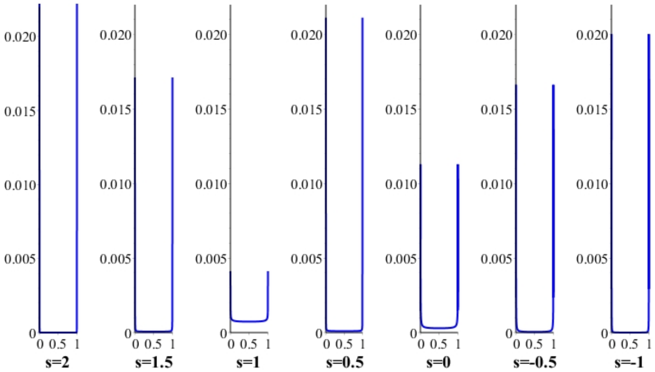
DDZ

Local Levels of Kolmogorov-Smirnov Tests



$\alpha_{i,n}^{+}$ for $\alpha = 0.05$ and $n = 100, 500, 1000$ and the corresponding asymptotic local levels (black curve)

Local Levels of Tests Based on φ-Divergences



$\alpha_{i,n} \equiv \alpha_{i,n}(s)$ with $\alpha = 0.05$ and $n = 1000$

$s = 2$: Higher Criticism (HC), $s = 1$: Berk-Jones (BJ),

$s = 0$: reversed BJ, $s = -1$: studentized HC

GOF Tests in Terms of Local Levels

For given critical values $c_{i,n}, \tilde{c}_{i,n}$, $i = 1, \ldots, n$, we get
$\alpha_{i,n}^{+} = \mathbb{P}(U_{i:n} \leq c_{i,n})$ and $\alpha_{i,n} = \mathbb{P}(U_{i:n} \leq c_{i,n}) + \mathbb{P}(U_{i:n} \geq \tilde{c}_{i,n})$.

$U_{i:n} \sim \mathcal{B}eta(i, n-i+1)$ with cdf $F_{i,n-i+1}$

One-sided test: $c_{i,n} = F_{i,n-i+1}^{-1}(\alpha_{i,n}^{+})$, $i = 1, \ldots, n$

Two-sided test: Split $\alpha_{i,n}$ in $\alpha_{i,n}^{(1)}$ and $\alpha_{i,n}^{(2)}$ such that
$\alpha_{i,n} = \alpha_{i,n}^{(1)} + \alpha_{i,n}^{(2)}$ (e.g., $\alpha_{i,n}^{(1)} = \alpha_{i,n}^{(2)} = \alpha_{i,n}/2$)
$\Rightarrow c_{i,n} = F_{i,n-i+1}^{-1}(\alpha_{i,n}^{(1)})$ and $\tilde{c}_{i,n} = F_{i,n-i+1}^{-1}(1 - \alpha_{i,n}^{(2)})$, $i = 1, \ldots, n$

GOF Test With Equal Local Levels

$$\boxed{\alpha^+_{1,n} = \ldots = \alpha^+_{n,n} = \alpha^{loc}_n \text{ or } \alpha_{1,n} = \ldots = \alpha_{n,n} = \alpha^{loc}_n}$$

$\varphi^+(\alpha^{loc}_n)$: $c_{i,n} = F^{-1}_{i,n-i+1}(\alpha^{loc}_n)$

$\varphi(\alpha^{loc}_n)$: $c_{i,n} = F^{-1}_{i,n-i+1}(\alpha^{loc}_n/2)$ and $\tilde{c}_{i,n} = 1 - F^{-1}_{i,n-i+1}(\alpha^{loc}_n/2)$

$p_{i,n} = F_{i,n-i+1}(X_{i:n})$, $i = 1, \ldots, n$, one-sided p-values

$$M^+_n = \min_{i=1,\ldots,n} p_{i,n} \text{ and } M_n = 2 \min_{i=1,\ldots,n} \{p_{i,n}, 1 - p_{i,n}\}$$

$$\varphi^+(\alpha^{loc}_n) = 1 \text{ iff } M^+_n \leq \alpha^{loc}_n \text{ and } \varphi(\alpha^{loc}_n) = 1 \text{ iff } M_n \leq \alpha^{loc}_n$$

$\Rightarrow$ $\varphi^+(\alpha^{loc}_n)$ and $\varphi(\alpha^{loc}_n)$ are minimum p-value (minP) GOF tests.

minP Tests: Different Names and Representations

- Berk & Jones [1978,1979] introduced M_n^+ and M_n as minimum level attained statistics: optimality and Bahadur efficiency;
- Buja & Rolke [2006] (unpublished): minP tests in terms of bounding functions;
- Gontscharuk, Landwehr & Finner (talks at MCP 2011 and MCP 2013): GOF tests with equal local levels, HC tests;
- Aldor et al. [2013]: tests based on the new tail-sensitive simultaneous confidence bands;
- Mary & Ferrari [2014]: non-asymptotic standardization of binomial counts, HC framework;
- Preprints: calibrated KS tests in Moskovich et al., Dirichlet-based tests in Kaplan & Goldman.

Level α minP GOF Tests

Find α_n^{loc} so that $\mathbb{P}(M_n^+ \leq \alpha_n^{loc} | H_0) = \alpha$ or $\mathbb{P}(M_n \leq \alpha_n^{loc} | H_0) = \alpha$.

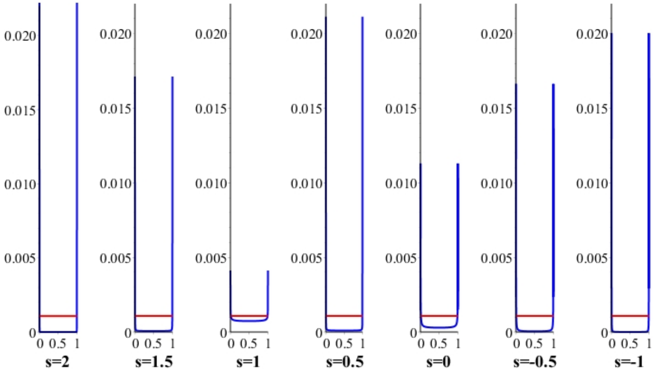
Let φ be an exact level α test with local levels $\alpha_{i,n}$. Then

$$\min_{i=1,\ldots,n} \alpha_{i,n} \leq \alpha_n^{loc} \leq \max_{i=1,\ldots,n} \alpha_{i,n}.$$

Example: $i \equiv i_n$ such that $i/n \to \zeta \in (0,1)$ leads to the asymptotic KS local level $\alpha_\zeta^{KS} = 1 - \Phi(-\log(\alpha)/(2\zeta(1-\zeta)))$. Hence,

$$\alpha_n^{loc} \leq 1 - \Phi(\sqrt{-2\log(\alpha)}) + o(1), \quad n \in \mathbb{N}.$$

GOF Tests Based on φ-Divergences



$\alpha_{i,n} \equiv \alpha_{i,n}(s)$ with $\alpha = 0.05$ and $n = 1000$,

$s = 2$: Higher Criticism, $s = 1$: Berk-Jones

Higher Criticism (HC) Statistics

- HC statistics are normalized KS statistics:

$$HC_n^+ = \max_{i=1,\dots,n} \sqrt{n}\frac{i/n - X_{i:n}}{\sqrt{X_{i:n}(1-X_{i:n})}},$$
$$HC_n = \max_{i=1,\dots,n} \left\{\sqrt{n}\frac{i/n - X_{i:n}}{\sqrt{X_{i:n}(1-X_{i:n})}}, \sqrt{n}\frac{X_{i:n} - (i-1)/n}{\sqrt{X_{i:n}(1-X_{i:n})}}\right\}.$$

- Eicker [1979] and Jaeschke [1979] provided a lot of asymptotic results.
- Local levels of one-sided HC asymptotic level α tests

$$\alpha_{i,n}^{HC}(\alpha) \approx \frac{-\log(1-\alpha)}{2\log_2(n)\log(n)}$$

for the most $i \in \{1,\dots,n\}$, cf. Gontscharuk et al.[2015]

HC local levels are asymptotically equal in the sensitivity range.

Asymptotics of minP GOF Tests

Theorem 1: (Gontscharuk & Finner [2015]) The minP test with critical value d_n is an asymptotic level α test iff

$$\lim_{n\to\infty} d_n/\alpha_n^* = 1 \text{ with } \alpha_n^* \equiv \alpha_n^*(\alpha) = -\frac{\log(1-\alpha)}{2\log_2(n)\log(n)}.$$

Remark:

- Critical values d_n related to asymptotic level α minP tests converge to 0 for $n \to \infty$.
- The asymptotic critical value is the same for one- and two-sided minP tests.

z-Transformed minP Statistics

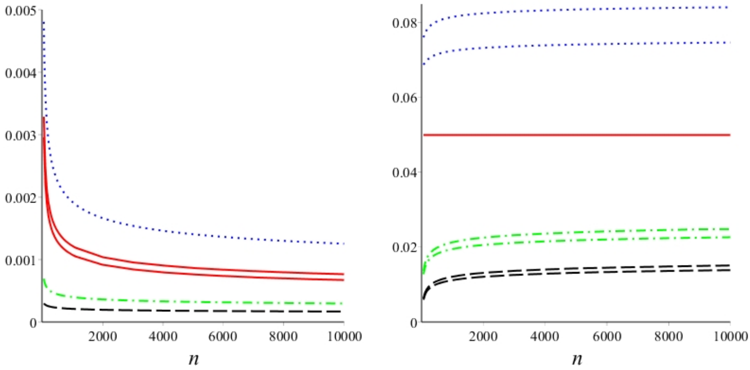
Theorem 2: (Gontscharuk & Finner [2015]) The z-transformed minP statistics $\Phi^{-1}(1-M_n^+)$ and $\Phi^{-1}(1-M_n/2)$ have the same asymptotic distribution as HC_n^+ and HC_n, resp.

Remark: Theorem 2 implies that the minP critical values

$$\alpha_n' = 1-\Phi(b_n(t_\alpha^+)) \quad \text{and} \quad \alpha_n'' = 2(1-\Phi(b_n(t_\alpha))),$$

where $b_n(t) = \sqrt{2\log_2(n)} + (\log_3(n) - \log(\pi) + 2t)/(2\sqrt{2\log_2(n)})$, $t_\alpha^+ = -\log(-\log(1-\alpha))$ and $t_\alpha = -\log(-\log(1-\alpha)/2)$, lead to asymptotic level α minP tests.

Applicability of Asymptotic Results



Left graph: α_n^*, α_n', α_n'', α_n^{loc} related to level α one-sided (upper curves) and two-sided (lower curves) minP tests.

Right graph: $\mathbb{P}(M_n^+ \leq d_n | H_0)$ (lower curves) and $\mathbb{P}(M_n \leq d_n | H_0)$ (upper curves) for $d_n = \alpha_n^*, \alpha_n', \alpha_n''$

Finite Approximation for α_n^{loc}

The asymptotic HC local levels are given by

$$\alpha_{i,n}^{HC}(\alpha) = \frac{-\log(1-\alpha)}{2\log_2(n)\log(n)}\left[1 + O\left(\frac{\log_3(n)}{\log_2(n)}\right)\right]$$

for $\log(n) \leq i \leq n - \log(n)$, cf. Gontscharuk et al. [2015].

Define

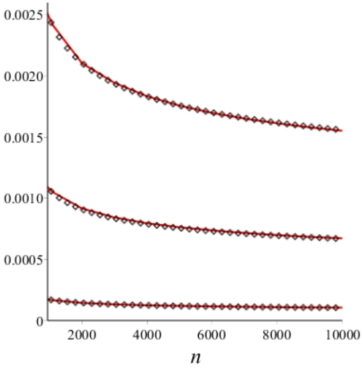
$$d_n(\alpha) = \frac{-\log(1-\alpha)}{2\log_2(n)\log(n)}\left[1 - c_\alpha \frac{\log_3(n)}{\log_2(n)}\right],$$

where $c_\alpha \in \mathbb{R}$ is a suitable constant.

Since $d_n(\alpha)/\alpha_n^* \to 1$ as $n \to \infty$, Theorem 1 implies that $d_n(\alpha)$ leads to asymptotic level α minP tests.

Approximated And Exact Critical Values

Critical values related to two-sided level α minP GOF tests



$d_n(\alpha)$ with $c_\alpha = 1.6, 1.3, 1.1$ (diamonds from bottom to top) and α_n^{loc} for $\alpha = 0.01, 0.05, 0.1$ (solid curves from bottom to top)

Simulated Global Levels Related to $d_n(\alpha)$

n	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
$n = 10^4$	0.00966 (0.00972)	0.04874 (0.04905)	0.09969 (0.09937)
$n = 5 \times 10^4$	0.00961	0.04971	0.10016
$n = 10^5$	0.01018	0.05019	0.10188
$n = 5 \times 10^5$	0.01001	0.05018	0.10115
$n = 10^6$	0.00973	0.04942	0.10135

$\mathbb{P}(M_n \leq d_n(\alpha)|H_0)$ (and $\mathbb{P}(M_n \leq \alpha_n^{loc}|H_0)$ for $n = 10^4$ only) simulated by 10^5 repetitions, where $d_n(\alpha)$ is based on $c_\alpha = 1.6, 1.3, 1.1$ for $\alpha = 0.01, 0.05, 0.1$, resp.

The minP GOF test works very well at least for considered α- and n-values.

Take Home Message

- Local levels can be seen as a measure of local sensitivity.
- One may construct a tailored GOF test by means of local levels.
- The minP test is a test with equal local levels.
- HC asymptotics is the key to the minP asymptotics.
- We provide three competing critical values leading to the asymptotic level α minP tests.
- The minP (as well as HC) asymptotics is very slow.
- New approximation for the minP critical value that works well for finite samples and leads to asymptotic level α tests.

References

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